CS1.404 (Spring 2025)
Optimization Methods
Deadline: 11.55 PM, May 7$^{\text{th}}$, 2025

Instructions

1. Attempting all questions is mandatory.
2. Typed submissions are not allowed. The final PDF should consist of handwritten pages.
3. Submit the final PDF on moodle titled RollNumber.pdf.

Question 1 (8 marks)

Consider the maximization problem

$$\max_{x_1,x_2} \; x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \quad \text{s.t.} \quad x_1 + x_2 = 1, \;\; x_1, x_2 \geq 0.$$

(i) Is the problem convex? (2)

(ii) Find all the KKT points of the problem. (3)

(iii) Find the optimal solution. (3)

Question 2 (8 marks)

Consider

$$\min_{x_1,x_2} \; x_1^2 + 2x_2^2 + x_1 \quad \text{s.t.} \quad x_1 + x_2 \leq a,$$

where $a \in \mathbb{R}$.

(i) Show for any a the solution is unique (without solving). (3)

(ii) Solve in terms of a. (2)

(iii) Let $f(a)$ be the optimal value. Write $f(a)$ explicitly and prove it is convex. (3)

Question 3 (10 marks)

Consider

$$(P) \quad \min_{x\in\mathbb{R}^n} \{\, a^T x : x^T Q x + 2b^T x + c \le 0\},$$

where $Q \in \mathbb{R}^{n\times n}$, $Q \succ 0$, $a \neq 0$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.

(i) For which Q, b, c is (P) feasible? (3)

(ii) For which Q, b, c are the KKT conditions necessary? (2)

(iii) For which Q, b, c are they sufficient? (2)

(iv) Under necessity, find the optimal solution via KKT. (3)

Question 4 (12 marks)

Use the KKT conditions to find an optimal solution for each of the following:

(i) $\min\limits_{x_1,x_2} 3x_1^2 + x_2^2,\ x_1 - x_2 + 8 \le 0,\ x_2 \ge 0.$ [2 marks]

(ii) $\min\limits_{x_1,x_2} 3x_1^2 + x_2^2,\ 3x_1^2 + x_2^2 + x_1 + x_2 + 0.1 \le 0,\ x_2 + 10 \ge 0.$ [2 marks]

(iii) $\min\limits_{x_1,x_2} 2x_1 + x_2,\ 4x_1^2 + x_2^2 - 2 \le 0,\ 4x_1 + x_2 + 3 \le 0.$ [2 marks]

(iv) $\min\limits_{x_1,x_2} x_1^3 + x_2^3,\ x_1^2 + x_2^2 \le 1.$ [3 marks]

(v) $\min\limits_{x_1,x_2} x_1^4 - x_2^2,\ x_1^2 + x_2^2 \le 1,\ 2x_2 + 1 \le 0.$ [3 marks]

Question 5 (5 marks)

Let $a > 0$. Find all solutions of

$$\max\{\, x_1 x_2 x_3 : a^2 x_1^2 + x_2^2 + x_3^2 \le 1\}.$$

[5 marks]

Question 6 (5 marks)

Consider

$$\min\ 3x_1^2 + x_1x_2 + 2x_2^2 \quad \text{s.t.} \quad 3x_1^2 + x_1x_2 + 2x_2^2 + x_1 - x_2 \ge 1,\ x_1 \ge 2x_2.$$

(i) Is the problem convex? (2)

(ii) Find the dual; is it convex? (3)

Question 7 (7 marks)

Consider

$$\min\ x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3 \quad \text{s.t.} \quad x_1 + x_2 + x_3 \le 1,\ x_3 \le 1.$$

(i) Is it convex? (2)

(ii) Find an optimal solution. (2)

(iii) Derive and solve the dual. (3)

Question 8 (8 marks)

Find the dual of

$$\min_{x>0} \sum_{i=1}^{n} (x_i \ln x_i - x_i) \quad \text{s.t.} \quad Ax \le b,$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$.

Question 9 (8 marks)

Find the dual of

$$\min \sum_{i=1}^{n} \left(a_i x_i^2 + 2b_i x_i + e^{\alpha_i x_i}\right) \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1,$$

where $a, \alpha \in \mathbb{R}^n_{++}$, $b \in \mathbb{R}^n$.

Question 10 (12 marks)

Find the dual of

$$\min \sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{\alpha_i}\right) + \|x\|^2 + 2a^T x \quad \text{s.t.} \quad x^T Ax + 2b^T x + c \le 0,$$

where $\alpha \in \mathbb{R}^n_{++}$, $A \succ 0, a, b \in \mathbb{R}^n, c \in \mathbb{R}$.

(i) Write down the dual. (6)

(ii) State an explicit data-based condition guaranteeing strong duality. (6)

Question 11 (17 marks)

Consider (with $0 \ln 0 = 0$)

$$\min\ x_1 + 2x_2 + 3x_3 + 4x_4 + \sum_{i=1}^{4} x_i \ln x_i \quad \text{s.t.} \quad \sum_{i=1}^{4} x_i = 1,\ x_i \geq 0.$$

(i) Show there cannot be more than one optimal solution. (3)

(ii) Find a dual with one multiplier. (4)

(iii) Solve that dual. (5)

(iv) Recover the primal optimal x. (5)