Constrained Optimization Problem

Ex:- SVM,

↓

Objective: Minimise$_{w,b,\xi_1 \ldots \xi_n}$ $\frac{1}{2}\|w\|^2 + c\sum_{i=1}^{n}\xi_i$

s.t. $\xi_i \geq 0 \quad i = 1 \ldots n$

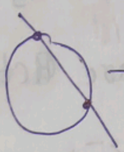
$y_i(\bar{w}^T\bar{x}_i + b) \geq 1 - \xi_i \quad i = 1 \ldots n$

2→ Script Set $X = \left\{ \bar{x} \in R^n \;\middle|\; \begin{array}{l} h_j(\bar{x}) \leq 0 \quad j = 1 \ldots l \\ \& \quad e_i(\bar{x}) = 0 \quad i = 1 \ldots m \end{array} \right\}$

Conditions to Satisfy:-

$e_i \rightarrow$ linear

SVM- Ex. of Constrained Convex Opt Prob



⟶ Not convex sets

If e_i not affine, const. set wont be convex

slide 7 ✓

- Descent directions may not be feasible

↓

Proof. By contradiction. Let $\bar{x}^*$ is a local minima.

Let there exists $\bar{d} \in \mathcal{F}(\bar{x}^*) \cap D(\bar{x}^*)$ and $\bar{d} \neq \bar{0}$

Because $\bar{d} \in \mathcal{F}(\bar{x}^*)$, $\exists\, \delta_1 > 0$, such that $\bar{x}^* + \alpha\bar{d} \in X$ $\forall \alpha \in (0, \delta_1)$

Because $\bar{d} \in D(\bar{x}^*)$, $\exists \delta_2 > 0$ such that

$$f(\bar{x}^* + \alpha \bar{d}) < f(\bar{x}^*)$$

$$\forall \alpha \in (0, \delta_2)$$

Let $\delta = \min(\delta_1, \delta_2)$

We see that, for δ,

$$f(\bar{x}^* + \alpha \bar{d}) < f(\bar{x}^*) \quad \forall \alpha \in (0, \delta)$$

and $\bar{x} + \alpha \bar{d} \in X_u \quad \forall \alpha \in (0, \delta)$

So, $f(\bar{x}) < f(\bar{x}^*) \quad \forall \bar{x} \in B(\bar{x}^*, \delta)$

which contradicts that $\bar{x}^*$ is local minima

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~~If all points~~

$x_A \rightarrow$ int. of $h_2 \cap h_3$

Both h_2, h_3 are Active.

$h_2(\bar{x}_A) = h_3(\bar{x}_A) = 0$.

$h_1, h_4, h_5 \rightarrow$ not active

Ex:- for $\bar{x}_A$, ~~h_2~~

$\bar{x}_B \rightarrow h_4$ - active

$h_1(\bar{x}_A) < 0$

no other active

Similarly for $\bar{x}_B$,

$\bar{x}_C \rightarrow$ no active

$h_1(\bar{x}_B) < 0 \quad h_3(\bar{x}_B) < 0 \quad h_5(\bar{x}_B) < 0$

$h_2(\bar{x}_B) < 0 \quad h_4(\bar{x}_B) = 0$

For $\bar{x}_c$, $h_i(\bar{x}_c) < 0 \quad \forall i = 1 .. 5$

$$A(\bar{x}_A) = \{2, 3\}$$

$$A(\bar{x}_B) = \{4\}$$

$$A(\bar{x}_c) = \phi$$

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$$\tilde{F}(\bar{x}) = \{ \bar{d} \in R^n \mid \exists\, \delta_1 > 0 \text{ such that } h_j(\bar{x} + \alpha \bar{d}) \leq 0 \quad \forall \alpha \in (0, \delta_1), \ \forall j \in \{1, \cdots l\} \}$$

Lemma: $\tilde{F}(\bar{x}) \subseteq F(\bar{x})$

Let $\bar{d} \in \tilde{F}(\bar{x})$. So,

$$\nabla h_j(\bar{x})^T \bar{d} < 0 \quad \forall j \in A(\bar{x})$$

Thus $\exists\, \delta_1 > 0$, such that

$$h_j(\bar{x} + \alpha \bar{d}) < h_j(\bar{x}) \quad \forall \alpha \in (0, \delta_1)$$

and $\forall j \in A(\bar{x})$.

But $h_j(\bar{x}) = 0 \quad \forall j \in A(\bar{x})$

$$\Rightarrow h_j(\bar{x} + \alpha \bar{d}) < 0 \quad \forall \alpha \in (0, \delta_1)$$

and $\forall j \in A(\bar{x})$.

Let $I(\bar{x}) = \{1, \cdots l\} \setminus A(\bar{x})$ is set of inactive constraints

$h_j(\bar{x}) < 0 \quad \forall j \in I(\bar{x})$.

wrt to inactive constraints,

$\bar{x}$ is interior point. So all

$\bar{d}$ are feasible wrt to inactive const.

So, $\exists \; \delta_2 > 0$, such that

$h_j(\bar{x} + \alpha \bar{d}) \leq 0 \quad \forall \alpha \in (0, \delta_2)$

and $\forall j \in I(\bar{x})$.

Take $\delta = \min(\delta_1, \delta_2)$. For this δ,

we have $h_j(\bar{x} + \alpha \bar{d}) \leq 0 \quad \forall \alpha \in (0, \delta)$

and for $j = 1 \ldots l$

By definition of feasible directions $\bar{d} \in F(\bar{x})$

Hence, $\tilde{F}(\bar{x}) \subseteq F(\bar{x})$.

(9) ⇒

(10)

(11)

Ex1:- ⇒

$$\nabla h_1(\bar{x}) = \begin{bmatrix} 3(x_1 + x_2 - 1)^2 \\ 3(x_1 + x_2 - 1)^2 \end{bmatrix}$$

$$\nabla h_1(\bar{x}_A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5 → Fritz-John Optimality Conditions

(11) Ex 2

$\nabla h_1(\bar{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad h_1(\bar{x}) = x_1 + x_2 - 1$

$\Rightarrow$ not because (last line)

$\Rightarrow$ not $\cup$, $\cap$ ✓

(2times)

Ex 3

$x_1 + x_2 = 1$

$x_1 + x_2 \leq 1$ and $x_1 + x_2 \geq 1$

$h_1(\bar{x}) = x_1 + x_2 - 1$

$h_2(\bar{x}) = -x_1 - x_2 + 1$

$d_1 + d_2 < 0$ and $d_1 + d_2 > 0$

(12) $\longleftrightarrow$

(13) Let $A =$ . . .

$\bar{x}^*$ is local minima $\Rightarrow$ $\tilde{F}(\bar{x}^*) \cap \tilde{D}(\bar{x}^*) = \phi$

$$\{ \bar{d} \mid \nabla f(\bar{x}^*)^T \bar{d} < 0 \} \cap \left\{ \begin{array}{l} \bar{d} \mid \nabla h_j(\bar{x}^*)^T d < 0 \\ \forall j \in A(\bar{x}^*) \end{array} \right\} = \phi$$

$$\Rightarrow \{ \bar{d} \mid \nabla f(\bar{x}^*)^T \bar{d} < 0 \text{ and } \nabla h_j(\bar{x}^*)^T \bar{d} < 0 \;\; \forall j \in A(\bar{x}^*) \} = \phi$$

$\Rightarrow$ Solution set of $A\bar{d} < 0$ is empty.

- Farkas lemma is theorem of alternatives